

# PLANE QUARTICS: THE MATRIX OF BITANGENTS

FRANCESCO DALLA PIAZZA, ALESSIO FIORENTINO, AND RICCARDO SALVATI MANNI

**ABSTRACT.** Aronhold's classical result states that a plane quartic can be recovered by the configuration of any Aronhold systems of bitangents, i.e. special 7-tuples of bitangents such that the six points at which any subtriple of bitangents touches the quartic do not lie on the same conic in the projective plane. Leham (cf. [L05]) proved that a smooth plane quartic can be explicitly reconstructed from its 28 bitangents; this result improved Aronhold's method of recovering the curve. In a 2011 paper [PSV11] Plaumann, Sturmfels and Vinzant introduced an eight by eight symmetric matrix parametrizing the bitangents of a nonsingular plane quartic. The starting point of their construction is Hesse's result for which every smooth quartic curve has exactly 36 equivalence classes of linear symmetric determinantal representations. In this paper we tackle the inverse problem, i.e. the construction of the bitangent matrix starting from the 28 bitangents of the plane quartic.

## 1. INTRODUCTION

It is classically known that the number of the bitangents to a non singular curve of degree  $d$  in the projective plane is given by the formula  $\frac{1}{2}d(d-2)(d^2-9)$ . The main properties of the bitangents have been deeply investigated by geometers since the late nineteenth century, particularly with reference to the first non trivial case, namely the case of degree 4. Aronhold's classical result states that a plane quartic can be recovered by the configuration of any of the 288 7-tuples of bitangents such that the six points at which any subtriple of bitangents touches the quartic do not lie on the same conic in the projective plane; these 7-tuples of bitangents are known as Aronhold systems. Caporaso and Sernesi proved in [CS03] that the general plane quartic is uniquely determined by its 28 bitangents; furthermore, they extended this result to general canonical curves of genus  $g \geq 4$  (cf. [CS03b]). In [L05] Leham proved that a non singular plane quartic can be reconstructed from its 28 bitangents, providing a method to derive an explicit formula for the curve. These results have improved Aronhold's method of recovering the curve, because the knowledge of both the bitangents and their contact points on the curve is needed to get the configuration of the Aronhold systems, whereas the sole configuration of the bitangents is enough to describe the geometry of the plane quartic. In a 2011 paper [PSV11] Plaumann, Sturmfels and Vinzant introduced an eight by eight symmetric matrix parametrizing the bitangents of a nonsingular plane quartic. The starting point of their construction is Hesse's result for which every smooth quartic curve has exactly 36 equivalence classes of linear symmetric determinantal representations. Each determinantal representation corresponds to three quadrics in  $\mathbb{P}^3$  intersecting in eight points. Once such a representation is chosen, the quartic can be described as the curve of the degenerate quadrics of the net generated by the three quadrics corresponding to the representation. Since each line of the net that joins two of the eight intersection points is a bitangent of the curve, this leads to define the eight by eight bitangent matrix.

We briefly recall this construction. We refer to [Do12] for details and historical notes. Let  $\mathbb{P}^3 = \mathbb{P}(W)$  where  $W$  is a 4-dimensional vector space, and let

$$X = \{x_1, \dots, x_8\} = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^3$$

be an unordered set of 8 distinct points in  $\mathbb{P}^3$ , which are complete intersection of three quadrics. If the net

$$\Lambda_X := |H^0(\mathbb{P}^3, \mathcal{I}_X(2))| = \langle Q_1, Q_2, Q_3 \rangle$$

contains no quadric of rank  $\leq 2$ , then  $X$  is called a *regular Cayley octad* and  $\Lambda_X$  is called a *regular net*. Now, let  $C_X = C \subset \Lambda_X$  be the curve of the degenerate quadrics of the regular net  $\Lambda_X$ . If we denote by  $\Delta$  the quartic hypersurface of singular quadrics in the linear system  $|H^0(\mathbb{P}^3, \mathcal{O}(2))| \cong \mathbb{P}^9$  of all the quadrics of  $\mathbb{P}^3$ , then  $C = \Lambda_X \cap \Delta$ , thus  $C$  is a plane quartic, which is called the *Hesse curve* of the net. After choosing a basis  $\{Q_1, Q_2, Q_3\}$  of  $H^0(\mathbb{P}^3, \mathcal{I}_X(2))$ , the net can be identified to  $\mathbb{P}^2$  by means of the homogeneous coordinates  $z = (z_1, z_2, z_3)$  as follows

$$(z_1, z_2, z_3) \longleftrightarrow Q(z) = z_1 Q_1 + z_2 Q_2 + z_3 Q_3$$

and the curve  $C$  has equation  $\det(A(z)) = 0$ , where  $A(z)$  is the symmetric  $4 \times 4$  matrix of the bilinear form associated with the generic quadric  $Q(z)$ . Note that the Hesse curve of a regular net of quadrics is nonsingular and  $\det(A(z)) = 0$  is one of its 36 determinantal representations [H55].

Denoting by  $L_{ij} = \langle x_i, x_j \rangle$  for each  $1 \leq i < j \leq 8$  the line joining the points  $x_i$  and  $x_j$  of the Cayley octad, we obtain a set of 28 lines in  $\mathbb{P}^3$ . These 28 lines  $L_{ij}$  in  $\mathbb{P}^3$  are in correspondence to the 28 bitangents of  $C$  in  $\mathbb{P}^2$ , as any of the equations:

$$(1.1) \quad x_i^t A(z) x_j = 0,$$

actually define one of the bitangents to the curve (cf. Proposition 5.2).

As for the Aronhold systems of  $C$ , a notable method to reconstruct one of them from the net  $\Lambda_X$  is provided by the Steiner embedding (see [DO88] for details). The singular points of the quadrics in  $\Lambda_X$  describe a curve  $\Gamma$  in  $\mathbb{P}^3$ , known as the *Steiner curve* of the net. As the net is regular, this curve turns out to be a smooth curve of degree 6. More precisely, there exists an even theta characteristic  $\theta$  such that the map  $f : C \rightarrow \mathbb{P}^3$  sending a point  $p \in C$  to the singular point of the corresponding quadric  $Q(p)$  in  $\mathbb{P}^3$ , is defined by the complete linear series  $|\omega_C + \theta|$ . In particular, the map  $f$  is an isomorphism on the image  $f(C) = \Gamma$ , and a bijection is defined between classes of regular nets of quadrics in  $\mathbb{P}^3$  up to projective equivalences and isomorphism classes of smooth curves of genus 3 associated with a fixed even theta characteristic. If an order  $x_1, \dots, x_8$  is chosen for the eight points of the regular Cayley octad  $X$  of the corresponding net  $\Lambda_X$ , the projection from the point  $x_8$  onto  $\mathbb{P}^2$  sends  $x_1, \dots, x_7$  to a set of points  $y_1, \dots, y_7$  and the Steiner curve  $\Gamma$  to a sextic with seven double points at  $y_1, \dots, y_7$ . The images of the exceptional curves blown up from these points  $y_1, \dots, y_7$  are the seven bitangents corresponding to the lines  $L_{i8}$  joining  $x_i$  and  $x_8$ ; this set of bitangents is actually an Aronhold system for the curve  $C$ .

The algorithm described in [PSV11] is meant to compute the matrix  $A(z)$  for a non singular plane quartic  $C$ , described by the equation:

$$f(z_1, z_2, z_3) = c_{400}z_1^4 + c_{310}z_1^3z_2 + c_{301}z_1^3z_3 + c_{220}z_1^2z_2^2 + c_{211}z_1^2z_2z_3 + \dots + c_{004}z_3^4,$$

where  $c_{ijk}$  are the 15 coefficients of the quartic. A determinantal representation of  $C$  is obtained in terms of 3 suitable  $4 \times 4$  symmetric matrices as follows:

$$f(x, y, z) = \det(z_1 A_1 + z_2 A_2 + z_3 A_3) \equiv \det(A(z)),$$

and  $A_1$ ,  $A_2$  and  $A_3$  are the matrices of the bilinear forms associated to three quadrics  $Q_1$ ,  $Q_2$  and  $Q_3$  in  $\mathbb{P}^3$ . The algorithm determines  $A(z)$  from the bitangents of the curve. The 28 bitangents  $b_m(\tau, z)$  of the plane quartic  $C$  correspond to the 28 gradients of the odd theta functions, as they are related to the first term of the Taylor expansion of the theta functions with odd characteristics  $\theta_m(\tau, z)$ . A  $4 \times 4$  matrix  $V$  is built after a choice of three bitangents among the  $\binom{28}{3} = 3276$  possible choices. If the triple of bitangents is not a subtriple of an Aronhold system, then the determinant of the resulting matrix  $V$  is identically null. Hence, the algorithm needs to start from a triple of bitangents that is contained in an Aronhold system; the number of these triples is 2016 and the corresponding triples of gradients of odd theta functions are called azygetic. The other  $3276 - 2016 = 1260$  triples are known as syzygetic. Once an azygetic triple is fixed, the matrix  $A(z)$  is given by the adjoint of  $V$  divided by  $f^2$ . These 2016 determinantal representations factorize into 36 equivalence classes (two representations  $A(z)$  and  $A'(z)$  are equivalent whenever they are conjugated under the action of  $\mathrm{GL}_4$ , i.e.  $A'(z) = U^t A(z) U$ , for  $U \in \mathrm{GL}_4$ ).

The bitangent matrix originates from  $A(z)$ . We can consider the  $8 \times 4$  matrix given by the coordinates of the eight points of the Cayley octad:

$$\mathbf{X} := \begin{pmatrix} x_{10} & x_{11} & x_{12} & x_{13} \\ x_{20} & x_{21} & x_{22} & x_{23} \\ \vdots & \vdots & \vdots & \vdots \\ x_{80} & x_{81} & x_{82} & x_{83} \end{pmatrix},$$

and the  $8 \times 8$  symmetric matrix:

$$(1.2) \quad L_X(z) := \mathbf{X} A(z) {}^t \mathbf{X}.$$

Clearly,  $L_X(z)$  is a matrix of rank 4 with zero entries on the main diagonal. The 28 entries of  $L_X(z)$  off the main diagonal are linear forms in  $z$  that define the bitangents of  $C$ , as seen in (1.1). Notice that the determinantal representations given by each of the  $\binom{8}{4} = 70$  principal  $4 \times 4$  minors of the matrix represent the same quartic and lie in the same equivalence class. Once a representation is determined, the other 35 inequivalent representations can be obtained by acting properly on the corresponding Cayley octad. Each of the  $2016 = 56 \cdot 36$  azygetic triples appears as a product of the corresponding bitangents in exactly one of the  $\binom{8}{3} = 56$  principal  $3 \times 3$  minor of one of the 36 inequivalent bitangent matrices; thus these minors are parametrized by azygetic triples.

In this paper we tackle the inverse problem, i.e. the construction of the bitangent matrix starting from the 28 bitangents of the plane quartic. We need to start by a suitable  $8 \times 8$  matrix. Since we are interested in azygetic triples in order to obtain the bitangent matrix, we resort to Aronhold systems to build such a matrix. For any fixed even characteristic, there exist eight corresponding Aronhold sets, see Section 3, ( $288 = 36 \cdot 8$ ), which are obtained by translation from a chosen one. These 7-tuples of odd characteristics and the even one will be the rows of the matrix, whose  $3 \times 3$  principal minors will contain 56 distinct azygetic triples ( $2016 = 36 \cdot 56$ ). Thus we will work with

the following bitangent matrix:

$$\mathcal{M} := \begin{pmatrix} 0 & b_{77} & b_{64} & b_{51} & b_{46} & b_{23} & b_{15} & b_{32} \\ b_{77} & 0 & b_{13} & b_{26} & b_{31} & b_{54} & b_{62} & b_{45} \\ b_{64} & b_{13} & 0 & b_{35} & b_{22} & b_{47} & b_{71} & b_{56} \\ b_{51} & b_{26} & b_{35} & 0 & b_{17} & b_{72} & b_{44} & b_{63} \\ b_{46} & b_{31} & b_{22} & b_{17} & 0 & b_{65} & b_{53} & b_{74} \\ b_{23} & b_{54} & b_{47} & b_{72} & b_{65} & 0 & b_{36} & b_{11} \\ b_{15} & b_{62} & b_{71} & b_{44} & b_{53} & b_{36} & 0 & b_{27} \\ b_{32} & b_{45} & b_{56} & b_{63} & b_{74} & b_{11} & b_{27} & 0 \end{pmatrix},$$

see Section 5 for the explanation of the meaning of the indices.

Notice that generally this matrix has rank eight, so one has to determine suitable coefficients  $c_{ij}$  in such a way that the matrix  $(c_{ij}b_{ij})$  has rank four. The aim of this paper is to determine uniquely such coefficients, up to congruences by diagonal matrices, once an even characteristic  $m$  and a compatible Aronhold set of characteristics (i.e a level two structure of the moduli space of principally polarized abelian varieties of genus 3) are given. As multilinear algebra techniques are not sufficient to determine these coefficients, we will also need to carefully use Riemann's relations and Jacobi's formula.

The coefficients will turn out to be modular functions holomorphic along the locus of the period matrices of smooth plane quartics (cf. Theorem 6.2). Any other inequivalent bitangent matrix will be obtained by changing the even characteristic and considering the eight corresponding Aronhold sets.

## 2. ACKNOWLEDGMENTS

We are grateful to Giorgio Ottaviani and Edoardo Sernesi for explaining to us the construction of the bitangent matrix and the beautiful geometry related to plane quartics. The first three sections of this paper have been strongly influenced by discussions that the senior author had with them. We are also grateful to Igor V. Dolgachev for pointing out inaccuracies in the first version of the paper.

## 3. ARONHOLD SYSTEMS.

In this section we introduce Aronhold systems of bitangents and Aronhold sets of theta characteristics and recall some basic facts about characteristics and the action of the symplectic group on them.

The next definition is of central importance in the geometry of plane quartics. Let  $C$  be a nonsingular plane quartic.

**Definition 3.1.** *A 7-tuple  $\{\ell_1, \dots, \ell_7\}$  of bitangent lines to  $C$  is called an Aronhold system of bitangents if for each triple  $\ell_i, \ell_j, \ell_k$  the six points of contact of  $\ell_i \cup \ell_j \cup \ell_k$  with  $C$  are not on a conic.*

Not all 7-tuples of bitangents are Aronhold systems. There are exactly 288 Aronhold systems among the  $\binom{28}{7}$  7-tuples of bitangents of  $C$  (for more details we refer to [Do12]). Denote by  $\theta_i$  the effective half-canonical divisor such that  $2\theta_i = C \cap \ell_i$  (i.e.  $\theta_i$ , or  $\mathcal{O}(\theta_i)$ , is an odd theta-characteristic).

The condition that  $\{\ell_1, \dots, \ell_7\}$  is an Aronhold system is equivalent to the condition that for each triple of pairwise distinct indices  $i, j, k$  we have

$$|2K - \theta_i - \theta_j - \theta_k| = \emptyset$$

and replacing  $2K$  by  $2\theta_i + 2\theta_j$  the condition is seen to be equivalent to the following:  $\theta_i + \theta_j - \theta_k$  is an even theta-characteristic for each  $i \neq j \neq k$ . This can be taken as another definition of Aronhold system (cf. also Definition 3.4).

**Definition 3.2.** An Aronhold set on a non-hyperelliptic curve  $C$  of genus 3 is a 7-tuple  $\{\theta_1, \dots, \theta_7\}$  of distinct odd theta-characteristics such that  $\theta_i + \theta_j - \theta_k$  is an even theta-characteristic for each triple of pairwise distinct indices  $i, j, k$ .

We have a purely combinatoric interpretation of the above description.

A characteristic  $m$  is a column vector in  $\mathbb{Z}^{2g}$ , with  $m'$  and  $m''$  as first and second entry vectors. If we set:

$$(3.1) \quad e(m) = (-1)^{t m' m''},$$

then  $m$  is called even or odd according as  $e(m) = 1$  or  $-1$ . For any triplet  $m_1, m_2, m_3$  of characteristics we set

$$(3.2) \quad e(m_1, m_2, m_3) = e(m_1)e(m_2)e(m_3)e(m_1 + m_2 + m_3).$$

A sequence  $m_1, \dots, m_r$  of characteristics is essentially independent if for any choice of an even number of indices between 1 and  $r$  the sum of the corresponding characteristics is not congruent to 0 mod 2.

The unique action of  $\Gamma_g = \text{Sp}(2g, \mathbb{Z})$  on the set of characteristics mod 2 keeping invariant (3.1), (3.2) and the condition of being essentially independent is defined by

$$\sigma \cdot m := \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} m' \\ m'' \end{pmatrix} + \begin{pmatrix} \text{diag}(C^t D) \\ \text{diag}(A^t B) \end{pmatrix}.$$

Henceforward, we often shall consider characteristics with 0 and 1 as entries. In this situation a special role is played by sequences of characteristics that form a fundamental system, defined as follows.

**Definition 3.3.** A sequence of  $2g + 2$  characteristics in  $\mathbb{F}_2^{2g}$  is a fundamental system if all triplets are azygetic, i.e

$$e(m_i, m_j, m_k) = -1,$$

for all indices  $1 \leq i < j < k \leq 2g + 2$ .

Fundamental systems exist and are all conjugate under an extension of  $\Gamma_g$  by translations, we refer to [Igu80] and [Fay79] for details. The number of odd characteristics in a fundamental system is always congruent to  $g \bmod 4$ . So when  $g = 3$  we have fundamental systems with 3 or 7 odd characteristics.

From now on we fix our attention on genus 3 case.

**Definition 3.4.** *Let*

$$m_0, n_1, n_2, \dots, n_7$$

*be a fundamental system with one even characteristic,  $m_0$ , and seven odd characteristics. In this case*

$$n_1, n_2, \dots, n_7$$

*is called an Aronhold set and necessarily  $m_0 = \sum_{i=1}^7 n_i$ .*

There are exactly  $288 = 36 \cdot 8$  such systems, hence each even characteristic appears exactly in eight such fundamental systems. We remark that the ordered set of fundamental systems are

$$288 \cdot 7! = 36 \cdot 8! = |\mathrm{Sp}(6, \mathbb{F}_2)|.$$

Concerning these fundamental systems with a fixed even characteristic  $m_0$ , we have the following Lemma.

**Lemma 3.5.** *A fundamental system  $m_0, n_1, n_2, \dots, n_7$  determines the remaining 7 via translations.*

*Proof.* The other seven fundamental systems can be obtained translating the initial fundamental system  $m_0, n_1, n_2, \dots, n_7$  with  $m_0 + n_i$  with  $i = 1, \dots, 7$ .  $\square$

**Remark 3.6.** *We observe that the  $8 \times 8$  matrix*

$$\begin{pmatrix} m_0 & n_1 & n_2 & \dots & n_7 \\ n_1 & m_0 & (m_0 + n_1) + n_2 & \dots & (m_0 + n_1) + n_7 \\ n_2 & (m_0 + n_2) + n_1 & m_0 & \dots & (m_0 + n_2) + n_7 \\ \dots & \dots & \dots & \dots & \dots \\ n_7 & (m_0 + n_7) + n_1 & \dots & \dots & m_0 \end{pmatrix}$$

*is symmetric in the symbols. It is unique once we fix a row, up to permutation of rows (and corresponding symmetric permutation of columns). Here is an explicit example (we use a row notation):*

$$\begin{pmatrix} [000, 000] & [111, 111] & [110, 100] & [101, 001] & [100, 110] & [010, 011] & [001, 101] & [011, 010] \\ [111, 111] & [000, 000] & [001, 011] & [010, 110] & [011, 001] & [101, 100] & [110, 010] & [100, 101] \\ [110, 100] & [001, 011] & [000, 000] & [011, 101] & [010, 010] & [100, 111] & [111, 001] & [101, 110] \\ [101, 001] & [010, 110] & [011, 101] & [000, 000] & [001, 111] & [111, 010] & [100, 100] & [110, 011] \\ [100, 110] & [011, 001] & [010, 010] & [001, 111] & [000, 000] & [110, 101] & [101, 011] & [111, 100] \\ [010, 011] & [101, 100] & [100, 111] & [111, 010] & [110, 101] & [000, 000] & [011, 110] & [001, 001] \\ [001, 101] & [110, 010] & [111, 001] & [100, 100] & [101, 011] & [011, 110] & [000, 000] & [010, 111] \\ [011, 010] & [100, 101] & [101, 110] & [110, 011] & [111, 100] & [001, 001] & [010, 111] & [000, 000] \end{pmatrix}.$$

*Notice that 36 essentially different such matrices can be constructed, corresponding to the 36 even characteristics.*

In the next section we will show how this matrix can be used to build the bitangent matrix by means of gradients of odd theta functions.

## 4. THETA FUNCTIONS

We intend to give an explicit analytic expression for the bitangents. The main tool will be theta functions. We denote by  $\mathcal{H}_g$  the *Siegel upper half-space*, i.e. the space of complex symmetric  $g \times g$  matrices with positive definite imaginary part. An element  $\tau \in \mathcal{H}_g$  is called a *period matrix*, and defines the complex abelian variety  $X_\tau := \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g$ . The group  $\Gamma_g := \mathrm{Sp}(2g, \mathbb{Z})$  acts on  $\mathcal{H}_g$  by automorphisms. For

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$$

the action is  $\gamma \cdot \tau := (a\tau + b)(c\tau + d)^{-1}$ . The quotient of  $\mathcal{H}_g$  by the action of the symplectic group is the moduli space of principally polarized abelian varieties (ppavs):  $\mathcal{A}_g := \mathcal{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$ . The case  $g = 1$  is special and in the following we will always assume  $g > 1$ .

We define the *level* subgroups of the symplectic group to be

$$\Gamma_g(n) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\},$$

$$\Gamma_g(n, 2n) := \left\{ \gamma \in \Gamma_g(n) \mid \mathrm{diag}(a^t b) \equiv \mathrm{diag}(c^t d) \equiv 0 \pmod{2n} \right\}.$$

The corresponding *level moduli spaces of ppavs* are denoted  $\mathcal{A}_g(n)$  and  $\mathcal{A}_g(n, 2n)$ , respectively.

A holomorphic function  $F : \mathcal{H}_g \rightarrow \mathbb{C}$  is called a *modular form of weight  $k$  with respect to  $\Gamma \subset \Gamma_g$*  if

$$F(\gamma \cdot \tau) = \det(c\tau + d)^k F(\tau), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \forall \tau \in \mathcal{H}_g.$$

More generally, let  $\rho : \mathrm{GL}(g, \mathbb{C}) \rightarrow \mathrm{End} V$  be some representation. Then a map  $F : \mathcal{H}_g \rightarrow V$  is called a  $\rho$ - or  $V$ -valued *modular form*, or, if there is no ambiguity in the choice of  $\rho$ , simply a *vector-valued modular form*, with respect to  $\Gamma \subset \Gamma_g$  when

$$F(\gamma \cdot \tau) = \rho(c\tau + d)F(\tau), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \forall \tau \in \mathcal{H}_g.$$

For  $m', m'' \in \mathbb{Z}^g$  and  $z \in \mathbb{C}^g$  we define the *theta function with characteristic  $m = \begin{bmatrix} m' \\ m'' \end{bmatrix}$*  to be

$$\theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z) := \theta_m(\tau, z) := \sum_{p \in \mathbb{Z}^g} \exp \pi i \left[ \left( p + \frac{m'}{2}, \tau \left( p + \frac{m'}{2} \right) \right) + 2 \left( p + \frac{m'}{2}, z + \frac{m''}{2} \right) \right],$$

where  $(\cdot, \cdot)$  at the exponent denotes the usual scalar product. Here we list some properties of the theta functions. First, we observe that

$$\theta_{m+2n}(\tau, z) = (-1)^{t m' n''} \theta_m(\tau, z), \quad n \in \mathbb{Z}^{2g}.$$

Hence, the theta functions with characteristics can be parametrized by  $2^{2g}$  vector columns  $m', m''$  with  $m'$  and  $m''$  thought as entries in  $\{0, 1\}^g$ . Note that these are the roots of the canonical bundle. The preceding formula is called *reduction formula*. Henceforward, we refer to such characteristics as *reduced characteristics* and to the corresponding theta functions as theta functions with half integral

characteristics; clearly all the properties stated in Section 3 also hold in this case. Then, we recall the behavior of the theta functions under a change of sign of the  $z$  variable:

$$\theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, -z) = e(m)\theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z).$$

The following formula shows that adding a so called *half period*,  $\tau \frac{m'}{2} + \frac{m''}{2}$ , to the argument  $z$  actually permutes the functions with half integral characteristics (see [Igu72] or [RF74]):

$$\theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z) = \exp\left(\pi i \left[ \left( \frac{m'}{2}, \tau \frac{m'}{2} \right) + 2 \left( \frac{m'}{2}, z + \frac{m''}{2} \right) \right]\right) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau, z + \tau \frac{m'}{2} + \frac{m''}{2}).$$

The *reduced characteristic*  $m$  is called *even* or *odd* depending on whether the scalar product  $m' \cdot m'' \in \mathbb{Z}_2$  is zero or one and the corresponding theta function is even or odd in  $z$ , respectively. The number of even (resp. odd) theta characteristics is  $2^{g-1}(2^g + 1)$  (resp.  $2^{g-1}(2^g - 1)$ ). The transformation law for theta functions under the action of the symplectic group is (see [Igu72]):

$$\theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z) = \phi(m', m'', \gamma, \tau, z) \det(c\tau + d)^{1/2} \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}({}^t\gamma^{-1} \begin{bmatrix} m' \\ m'' \end{bmatrix})(\gamma \cdot \tau, (c\tau + d)z),$$

where  $\phi$  is some complicated explicit function, and the action of  ${}^t\gamma^{-1}$  on characteristics is taken modulo integers. It is further known (see [Igu72], [SM94]) that for  $\gamma \in \Gamma_g(4, 8)$  we have  $\phi|_{z=0} = 1$ , while  ${}^t\gamma^{-1}$  acts trivially on the characteristics  $m$ . Thus the values of theta functions at  $z = 0$ , called *theta constants*, are modular forms of weight one half with respect to  $\Gamma_g(4, 8)$ . We will denote them with  $\theta_m$ .

The group  $\Gamma_g(2)/\Gamma_g(4, 8)$  acts on the set of theta-constants  $\theta_m$  by certain characters whose values are fourth roots of the unity, as shown in [SM94]. The action of  $\Gamma_g/\Gamma_g(2)$  on the set of theta with half integral characteristics is by permutations. Since the group  $\Gamma_g(1, 2)$  fixes the null characteristic, it acts on  $\theta_0$  by a multiplier.

All odd theta constants with half integral characteristics vanish identically, as the corresponding theta functions are odd functions of  $z$ , and thus there are  $2^{g-1}(2^g + 1)$  non-trivial theta constants.

Differentiating the theta transformation law above with respect to different  $z_i$  and then evaluating at  $z = 0$ , we see that for  $\gamma \in \Gamma_g(4, 8)$  and  $m = \begin{bmatrix} m' \\ m'' \end{bmatrix}$  odd

$$\frac{\partial}{\partial z_i} \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z)|_{z=0} = \det(c\tau + d)^{1/2} \sum_j (c\tau + d)_{ij} \frac{\partial}{\partial z_j} \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\gamma \cdot \tau, (c\tau + d)z)|_{z=0},$$

in other words the gradient vector  $\text{grad}_z \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, 0)$  is a  $\mathbb{C}^g$ -valued modular form with respect to  $\Gamma_g(4, 8)$  under the representation  $\rho(M) = (\det M)^{1/2} M$ , for  $M \in \Gamma_g(4, 8)$ .

The set of all even theta constants defines the map

$$I\!P\!Th : \mathcal{A}_g(4, 8) \rightarrow \mathbb{P}^{2^{g-1}(2^g+1)-1}, \quad \bar{\tau} \mapsto [\cdots, \theta_m(\tau), \cdots],$$

with  $\bar{\tau} \in A_g(4, 8) = \mathcal{H}_g/\Gamma_g(4, 8)$  and  $\tau$  a representative of the equivalence class  $\bar{\tau}$ . It is known that the map  $I\!P\!Th$  is injective, see [Igu72] and references therein. Considering the set of gradients of all



odd theta functions at zero gives the map

$$\text{grTh} : \mathcal{H}_g \rightarrow (\mathbb{C}^g)^{\times 2^{g-1}(2^g-1)}, \quad \tau \mapsto \text{grTh}(\tau) := \left\{ \cdots, \text{grad}_z \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}, \cdots \right\}_{\text{all odd } m},$$

which due to modular properties descends to the quotient

$$\mathbb{P}\text{grTh} : \mathcal{A}_g(4, 8) \rightarrow (\mathbb{C}^g)^{\times 2^{g-1}(2^g-1)} / \rho(\text{GL}(g, \mathbb{C})),$$

where  $\text{GL}(g, \mathbb{C})$  acts simultaneously on all  $\mathbb{C}^g$ 's in the product by  $\rho$ .

The image of  $\mathbb{P}\text{grTh}$  actually lies in the Grassmannian,

$$\mathbb{P}\text{grTh} : \mathcal{A}_g(4, 8) \rightarrow \text{Gr}_{\mathbb{C}}(g, 2^{g-1}(2^g - 1))$$

of  $g$ -dimensional subspaces in  $\mathbb{C}^{2^{g-1}(2^g-1)}$ . The Plücker coordinates of this map are modular forms of weight  $\frac{g}{2} + 1$  and have been extensively studied, see [Igu83, SM83, GSM04]. Moreover, in [GSM04], it is implicitly proved the following proposition.

**Proposition 4.1.** *The map*

$$\mathbb{P}\text{grTh} : \mathcal{A}_3(4, 8) \rightarrow \text{Gr}_{\mathbb{C}}(3, 28)$$

*is injective.*

In genus 3 case the evaluation at zero of  $\theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z)$  and of the gradients have a significative meaning. In fact, as consequence of Riemann singularity theorem the following proposition holds.

**Proposition 4.2.** *Let  $\tau$  be a jacobian matrix, then it is the period matrix of a hyperelliptic jacobian if and only if there exist an even characteristic  $m$  with  $\theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, 0) = 0$ .*

*Let  $\tau$  be the period matrix of a non-hyperelliptic jacobian (i.e the jacobian of a plane quartic), then for all odd characteristics  $m$  the gradient vector  $\text{grad}_z \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, 0)$  parametrizes the bitangents of the plane quartic.*

**Remark 4.3.** *The equations of the bitangents are*

$$b_m(\tau, z) := \frac{\partial \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z)}{\partial z_1} \Big|_{z=0z_1} + \frac{\partial \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z)}{\partial z_2} \Big|_{z=0z_2} + \frac{\partial \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z)}{\partial z_3} \Big|_{z=0z_3} = 0.$$

*From now on, if it will be necessary, we will identify the gradient vectors with the bitangents*

The following corollary is easily derived.

**Corollary 4.4.** *The hyperelliptic locus  $\mathcal{I}_3 \subset \mathcal{A}_3$  is defined by the equation*

$$\prod_{m \text{ even}} \theta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, 0) = 0.$$

## 5. THE BITANGENT MATRIX

This section will be entirely devoted to the description of the bitangent matrix related to the plane quartic. First of all, we will see how the language introduced in the previous section suitably translates the combinatorics described in Section 3 so as to let us build the matrix. Then, we will recall the properties of such a matrix and how it completely describes the geometry of the curve.

Using the language of Section 4, the geometric condition defining an Aronhold system can be rephrased as a combinatorial condition, as  $\theta_i + \theta_j - \theta_k$  is an even theta characteristic whenever  $m_i + m_j + m_k$  is an even characteristic. So we can apply what we stated in Remark 3.6 to the  $8 \times 8$  symmetric matrix  $L_X(z)$ , cf. (1.2), since it is related to the matrix obtained using the derivatives of odd theta functions ordered as in Remark 3.6. More precisely, using the notations in Remark 3.6 we set

$$M_X(z) := \begin{pmatrix} 0 & b_{n_1}(\tau, z) & \dots & b_{n_7}(\tau, z) \\ b_{n_1}(\tau, z) & 0 & \dots & b_{n_7+m_0+n_1}(\tau, z) \\ \dots & \dots & 0 & \dots \\ b_{n_7}(\tau, z) & b_{n_1+m_0+n_7}(\tau, z) & \dots & 0 \end{pmatrix},$$

and

$$M_i := \begin{pmatrix} 0 & \frac{\partial}{\partial z_i} \theta_{n_1}(\tau, z)|_{z=0} & \dots & \frac{\partial}{\partial z_i} \theta_{n_7}(\tau, z)|_{z=0} \\ \frac{\partial}{\partial z_i} \theta_{n_1}(\tau, z)|_{z=0} & 0 & \dots & \frac{\partial}{\partial z_i} \theta_{n_7+m_0+n_1}(\tau, z)|_{z=0} \\ \dots & \dots & 0 & \dots \\ \frac{\partial}{\partial z_i} \theta_{n_7}(\tau, z)|_{z=0} & \frac{\partial}{\partial z_i} \theta_{n_1+m_0+n_7}(\tau, z)|_{z=0} & \dots & 0 \end{pmatrix}.$$

Thus the matrices

$$M_X(z) = z_1 M_1 + z_2 M_2 + z_3 M_3$$

and  $L_X(z)$  are related. In both cases the entries of the matrices  $M_X(z)$  and  $L_X(z)$  are the bitangents to the canonical curves, but they are not uniquely determined, (entries can differ by different proportionality factors) so the matrix  $M_X(z)$  has not necessarily rank 4.

We will manipulate the matrix  $M_X(z)$  and determine suitable coefficients  $c_{ij}$  in order that the matrix  $(c_{ij} M_X(z)_{ij})$  has rank four. We will do it using the action of the symplectic group, Riemann theta formula and Jacobi's derivative formula. We recall shortly them in the genus 3 case.

For any triple  $n_1, n_2, n_3$  of odd characteristics we set

$$D(n_1, n_2, n_3)(\tau) := \text{grad}_z \theta_{n_1}(\tau, 0) \wedge \text{grad}_z \theta_{n_2}(\tau, 0) \wedge \text{grad}_z \theta_{n_3}(\tau, 0).$$

We recall when such nullwerte of jacobian of theta functions is a polynomial in the theta constants. The following statement holds, see [Igu81].

**Proposition 5.1.** *Suppose that  $g = 3$  and  $n_1, n_2, n_3$  are odd characteristics distinct mod 2; then  $D(n_1, n_2, n_3)$  is a polynomial in the theta constants if and only if  $n_1, n_2, n_3$  form an azygetic triplet. Moreover, we have*

$$D(n_1, n_2, n_3)(\tau) = \pm(\pi)^3 \theta_{m_1} \theta_{m_2} \theta_{m_3} \theta_{m_4} \theta_{m_5}(\tau),$$

with  $m_1, \dots, m_5$  even characteristics and  $n_1, n_2, n_3, m_1, \dots, m_5$  a uniquely determined fundamental system of characteristics.

Also, we briefly recall that Riemann's quartic addition theorem for theta constants with characteristic in genus three has the form

$$r_1 = r_2 + r_3,$$

where each  $r_i$  is a product of four theta constants with characteristics forming an even coset of a two-dimensional isotropic space. Such isotropic spaces are constructed by means of the symplectic form on  $\mathbb{F}_2^6$  defined by

$$e(m, n) := (-1)^{m'^t n'' - m''^t n'}.$$

A full description of the curve  $C$  is provided by the matrix  $L_X(z)$  defined in (1.2), as the following notable proposition states.

**Proposition 5.2.** *Let  $L_X(z)$  be the matrix defined in (1.2), then:*

- (i) *given the net  $\Lambda_X$  and a basis  $\{Q_1, Q_2, Q_3\}$  of it, the matrix  $L_X(z)$  is uniquely defined up to simultaneous multiplication by a constant factor of a row and the corresponding column and up to simultaneous permutations of rows and columns;*
- (ii) *the 28 entries of  $L_X(z)$  outside the main diagonal are linear forms in  $z$  that define the bitangents of  $C$ ;*
- (iii) *the seven bitangents on a given row (column) are elements of an Aronhold system. The 8 Aronhold systems represented by the rows (columns) of  $L_X(z)$  are associated to the even theta characteristic on  $C$  defined by the net  $\Lambda_X$ ;*
- (iv)  *$L_X(z)$  has identically rank 4, and any of its  $4 \times 4$  minors is a polynomial of degree 4 in  $z$  which defines  $C$ .*

*Proof.*

- (i) A change of the order of the eight points of the Cayley octad  $x_1, \dots, x_8$  corresponds to a simultaneous permutation of rows and columns for the matrix. A change of the homogeneous coordinates of these points corresponds to a simultaneous multiplication by a constant factor. Hence, the statement follows.
- (ii) We refer to [PSV11] for details (cf. Section 1, eq. (1.3)).
- (iii) The statement is a consequence of what we explained at the beginning of this section.
- (iv) The claim obviously follows by the way the matrix was defined.

□

## 6. DETERMINING ANALYTICALLY THE BITANGENT MATRIX

Now we want to determine analytically the bitangent matrix; this will be a partial converse of the Proposition 5.2.

Our initial datum will be the 28 gradients of odd theta functions evaluated at 0, corresponding to the bitangents and a chosen even characteristics  $m$ . Therefore, to obtain a matrix congruent to  $L_X(z)$ ,

we have to determine the values of the functions  $c_{n_i}(\tau)$  in the matrix:

$$\begin{pmatrix} 0 & c_{n_1}(\tau)b_{n_1}(\tau, z) & c_{n_2}(\tau)b_{n_2}(\tau, z) & \dots & c_{n_7}(\tau)b_{n_7}(\tau, z) \\ c_{n_1}(\tau)b_{n_1}(\tau, z) & 0 & c_{m+n_1+n_2}(\tau)b_{m+n_1+n_2}(\tau, z) & \dots & c_{m+n_1+n_7}(\tau)b_{m+n_1+n_7}(\tau, z) \\ c_{n_2}(\tau)b_{n_2}(\tau, z) & c_{m+n_1+n_2}(\tau)b_{m+n_1+n_2}(\tau, z) & 0 & \dots & c_{m+n_2+n_7}(\tau)b_{m+n_2+n_7}(\tau, z) \\ \dots & \dots & \dots & \dots & \dots \\ c_{n_7}(\tau)b_{n_7}(\tau, z) & c_{m+n_1+n_7}(\tau)b_{m+n_1+n_7}(\tau, z) & \dots & \dots & 0 \end{pmatrix}.$$

There are, up to permutations, 8 Aronhold sets whose sum is the given characteristic. The subgroup of the symplectic group fixing the characteristic  $m$ , permutes the element of a fixed Aronhold set and the eight Aronhold sets. Now we choose an even characteristic and an Aronhold set. As we wrote, the number of possibilities is exactly equal to  $|Sp(6, \mathbb{F}_2)|$ . To simplify our computation we assume  $m = 0$  and as Aronhold set we use the one in Remark 3.6.

We want to obtain a matrix of rank four. Because of the action of the symplectic group, we can assume that all columns are linear combination of the first four. This will be our starting hypothesis. We recall that any set of four bitangents coming from an Aronhold set form a fundamental system of  $\mathbb{P}^2$ . We need seven bitangents  $b_1, \dots, b_7$  forming an Aronhold set. We choose those in Section 3. Let  $\mathcal{M}$  be the symmetric matrix:

$$\mathcal{M} := \begin{pmatrix} 0 & b_{77} & b_{64} & b_{51} & b_{46} & b_{23} & b_{15} & b_{32} \\ b_{77} & 0 & b_{13} & b_{26} & b_{31} & b_{54} & b_{62} & b_{45} \\ b_{64} & b_{13} & 0 & b_{35} & b_{22} & b_{47} & b_{71} & b_{56} \\ b_{51} & b_{26} & b_{35} & 0 & b_{17} & b_{72} & b_{44} & b_{63} \\ b_{46} & b_{31} & b_{22} & b_{17} & 0 & b_{65} & b_{53} & b_{74} \\ b_{23} & b_{54} & b_{47} & b_{72} & b_{65} & 0 & b_{36} & b_{11} \\ b_{15} & b_{62} & b_{71} & b_{44} & b_{53} & b_{36} & 0 & b_{27} \\ b_{32} & b_{45} & b_{56} & b_{63} & b_{74} & b_{11} & b_{27} & 0 \end{pmatrix},$$

where  $b_{ij} := \sum_{k=1}^3 (\frac{\partial}{\partial z_k} \theta_{ij}|_{z=0}) z_k$ ,  $\theta_{ij}$  being the theta function associated with the odd characteristic  $[i] := \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ , where  $i = a_1 2^2 + a_2 2 + a_3$  and  $j = b_1 2^2 + b_2 2 + b_3$ . Each equation  $b_{ij} = 0$  defines a bitangent. Clearly these bitangents do not change by multiplying each entry of the matrix by a function which does not depend on the variables  $z_k$ , with  $k = 1, 2, 3$ .

To find out what these 28 coefficients are, we will resort to the following procedure. We will first determine the coefficients of suitable  $5 \times 5$  principal minors so as to get symmetric matrices of rank 4; this will define some relations among the column vectors of the matrix. Then we will act on these minors properly so as to make their common entries equal. Finally we will use the resulting relations among the column vectors to determine the remaining coefficients of the matrix.

We first focus on the submatrix obtained by taking the first five rows and the first five columns.

We need to compute  $\lambda_{ij}$  such that:

$$(6.1) \quad \text{rank} \begin{pmatrix} 0 & \lambda_{77}b_{77} & \lambda_{64}b_{64} & \lambda_{51}b_{51} & \lambda_{46}b_{46} \\ \lambda_{77}b_{77} & 0 & \lambda_{13}b_{13} & \lambda_{26}b_{26} & \lambda_{31}b_{31} \\ \lambda_{64}b_{64} & \lambda_{13}b_{13} & 0 & \lambda_{35}b_{35} & \lambda_{22}b_{22} \\ \lambda_{51}b_{51} & \lambda_{26}b_{26} & \lambda_{35}b_{35} & 0 & \lambda_{17}b_{17} \\ \lambda_{46}b_{46} & \lambda_{31}b_{31} & \lambda_{22}b_{22} & \lambda_{17}b_{17} & 0 \end{pmatrix} = 4.$$

Note that  $\text{rk}(D^tAD) = \text{rk}(A)$  for any invertible diagonal matrix  $D$ . Therefore, the matrix in (6.1) is determined up to an invertible diagonal matrix which acts by congruence.

The condition of linear dependence on the vector columns  $V_i$  of the matrix in (6.1):

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 + \alpha_5 V_5 = 0$$

can be turned into:

$$(6.2) \quad \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3 + \tilde{V}_4 - \tilde{V}_5 = 0$$

when both the sides of the matrix are multiplied by the diagonal matrix  $\text{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}, \alpha_4^{-1}, -\alpha_5^{-1})$ . Hence, we can compute the coefficients  $\lambda_{ij}$  by demanding the condition (6.2) without any loss of generality. Note that whenever such an operation is performed again on the matrix, the diagonal matrix on the left will change the coefficients in (6.2).

On the first row (6.2) leads to:

$$\lambda_{77}b_{77} + \lambda_{64}b_{64} + \lambda_{51}b_{51} = \lambda_{46}b_{46},$$

which is equivalent to a linear system of three equations in the variables  $\lambda_{77}$ ,  $\lambda_{64}$ ,  $\lambda_{51}$ ,  $\lambda_{46}$ :

$$(6.3) \quad \begin{cases} \lambda_{77}\partial_1\theta_{77}|_{z=0} + \lambda_{64}\partial_1\theta_{64}|_{z=0} + \lambda_{51}\partial_1\theta_{51}|_{z=0} = \lambda_{46}\partial_1\theta_{46}|_{z=0} \\ \lambda_{77}\partial_2\theta_{77}|_{z=0} + \lambda_{64}\partial_2\theta_{64}|_{z=0} + \lambda_{51}\partial_2\theta_{51}|_{z=0} = \lambda_{46}\partial_2\theta_{46}|_{z=0} \\ \lambda_{77}\partial_3\theta_{77}|_{z=0} + \lambda_{64}\partial_3\theta_{64}|_{z=0} + \lambda_{51}\partial_3\theta_{51}|_{z=0} = \lambda_{46}\partial_3\theta_{46}|_{z=0} \end{cases} \quad \forall \tau \in \mathcal{H}_3,$$

where  $\partial_k\theta_{ij}|_{z=0} := \frac{\partial}{\partial z_k}\theta_{ij}|_{z=0}$ , with  $k = 1, 2, 3$ . The solution of (6.3) can be determined up to a constant:

$$\lambda_{77} = D(46, 64, 51), \quad \lambda_{64} = D(77, 64, 46), \quad \lambda_{51} = D(77, 46, 51), \quad \lambda_{46} = D(77, 64, 51).$$

Here, as in Section 5,  $D(l, m, n) := \det \frac{\partial(\theta_1, \theta_m, \theta_n)}{\partial z_1 \partial z_2 \partial z_3}$ . By repeating this procedure on each row we get the matrix:

$$M = \begin{pmatrix} 0 & D(46, 64, 51)b_{77} & D(77, 46, 51)b_{64} & D(77, 64, 46)b_{51} & D(77, 64, 51)b_{46} \\ D(31, 13, 26)b_{77} & 0 & D(77, 31, 26)b_{13} & D(77, 13, 31)b_{26} & D(77, 13, 26)b_{31} \\ D(22, 13, 35)b_{64} & D(64, 22, 35)b_{13} & 0 & D(64, 13, 22)b_{35} & D(64, 13, 35)b_{22} \\ D(17, 26, 35)b_{51} & D(51, 17, 35)b_{26} & D(51, 26, 17)b_{35} & 0 & D(51, 26, 35)b_{17} \\ D(17, 31, 22)b_{46} & D(46, 17, 22)b_{31} & D(46, 31, 17)b_{22} & D(46, 31, 22)b_{17} & 0 \end{pmatrix}.$$

Although this matrix is not symmetric, it can be turned into a symmetric one by multiplying it on

the left by a suitable diagonal matrix (note that this operation does not change the rank). If we choose the matrix  $D_1$ :

$$D_1 := \text{diag} \left( 1, \frac{D(46, 64, 51)}{D(31, 13, 26)}, \frac{D(77, 46, 51)}{D(22, 13, 35)}, \frac{D(77, 64, 46)}{D(17, 26, 35)}, \frac{D(77, 64, 51)}{D(17, 31, 22)} \right),$$

we set  $S'_1 := D_1 M$  and we get:

$$S'_1 = \begin{pmatrix} 0 & D(46, 64, 51)b_{77} & D(77, 46, 51)b_{64} & D(77, 64, 46)b_{51} & D(77, 64, 51)b_{46} \\ D(46, 64, 51)b_{77} & 0 & \frac{D(46, 64, 51)D(77, 31, 26)}{D(31, 13, 26)}b_{13} & \frac{D(46, 64, 51)D(77, 13, 31)}{D(31, 13, 26)}b_{26} & \frac{D(46, 64, 51)D(77, 13, 26)}{D(31, 13, 26)}b_{31} \\ D(77, 46, 51)b_{64} & \frac{D(77, 46, 51)D(64, 22, 35)}{D(22, 13, 35)}b_{13} & 0 & \frac{D(77, 46, 51)D(64, 13, 22)}{D(22, 13, 35)}b_{35} & \frac{D(77, 46, 51)D(64, 13, 35)}{D(22, 13, 35)}b_{22} \\ D(77, 64, 46)b_{51} & \frac{D(77, 64, 46)D(51, 17, 35)}{D(17, 26, 35)}b_{26} & \frac{D(77, 64, 46)D(51, 26, 17)}{D(17, 26, 35)}b_{35} & 0 & \frac{D(77, 64, 46)D(51, 26, 35)}{D(17, 26, 35)}b_{17} \\ D(77, 64, 51)b_{46} & \frac{D(77, 64, 51)D(46, 17, 22)}{D(17, 31, 22)}b_{31} & \frac{D(77, 64, 51)D(46, 31, 17)}{D(17, 31, 22)}b_{22} & \frac{D(77, 64, 51)D(46, 31, 22)}{D(17, 31, 22)}b_{17} & 0 \end{pmatrix}.$$

Thanks to the relations among the determinants induced by Jacobi's derivative formula [Igu80], the matrix  $S'_1$  is easily seen to be symmetric. Note that a different diagonal matrix  $D_i$  can be chosen for this operation in such a way that the matrix  $S'_i := D_i M$  and the matrix  $M$  have the same entries on the  $i$ -th row. A straightforward computation proves that  $D_i D_1^{-1} = c_i Id$  with a suitable  $c_i$ , hence  $S'_i = c_i S'_1$ .

We can get a more convenient form for  $S'_1$  acting by congruence with the diagonal matrix:

$$T_1 := \text{diag} \left( 1, \frac{D(31, 13, 26)}{D(46, 64, 51)D(77, 31, 26)}, \frac{D(22, 13, 35)}{D(77, 46, 51)}, 1, 1 \right).$$

Then we have:

$$S_1 := T_1 S'_1 T_1 = \begin{pmatrix} 0 & \frac{D(31, 13, 26)}{D(77, 31, 26)}b_{77} & D(22, 13, 35)b_{64} & D(77, 64, 46)b_{51} & D(77, 64, 51)b_{46} \\ \frac{D(31, 13, 26)}{D(77, 31, 26)}b_{77} & 0 & \frac{D(22, 13, 35)}{D(77, 46, 51)}b_{13} & \frac{D(77, 13, 31)}{D(77, 31, 26)}b_{26} & \frac{D(77, 13, 26)}{D(77, 31, 26)}b_{31} \\ D(22, 13, 35)b_{64} & \frac{D(22, 13, 35)}{D(77, 46, 51)}b_{13} & 0 & D(64, 13, 22)b_{35} & D(64, 13, 35)b_{22} \\ D(77, 64, 46)b_{51} & \frac{D(77, 13, 31)}{D(77, 31, 26)}b_{26} & D(64, 13, 22)b_{35} & 0 & \frac{D(77, 64, 46)D(51, 26, 35)}{D(17, 26, 35)}b_{17} \\ D(77, 64, 51)b_{46} & \frac{D(77, 13, 26)}{D(77, 31, 26)}b_{31} & D(64, 13, 35)b_{22} & \frac{D(77, 64, 46)D(51, 26, 35)}{D(17, 26, 35)}b_{17} & 0 \end{pmatrix}.$$

Likewise, the whole procedure can be repeated for the submatrices of  $\mathcal{M}$  obtained by replacing the fifth column and row respectively with the sixth, the seventh and the eighth. Then we get the following symmetric matrices of rank 4:

$$S_2 := \begin{pmatrix} 0 & \frac{D(54, 13, 26)}{D(77, 54, 26)}b_{77} & D(47, 13, 35)b_{64} & D(77, 64, 23)b_{51} & D(77, 64, 51)b_{23} \\ \frac{D(54, 13, 26)}{D(77, 54, 26)}b_{77} & 0 & \frac{D(47, 13, 35)}{D(77, 23, 51)}b_{13} & \frac{D(77, 13, 54)}{D(77, 54, 26)}b_{26} & \frac{D(77, 13, 26)}{D(77, 54, 26)}b_{54} \\ D(47, 13, 35)b_{64} & \frac{D(47, 13, 35)}{D(77, 23, 51)}b_{13} & 0 & D(64, 13, 47)b_{35} & D(64, 13, 35)b_{47} \\ D(77, 64, 23)b_{51} & \frac{D(77, 13, 54)}{D(54, 13, 26)}b_{26} & D(64, 13, 47)b_{35} & 0 & \frac{D(77, 64, 23)D(51, 26, 35)}{D(72, 26, 35)}b_{72} \\ D(77, 64, 51)b_{23} & \frac{D(77, 13, 26)}{D(54, 13, 26)}b_{54} & D(64, 13, 35)b_{47} & \frac{D(77, 64, 23)D(51, 26, 35)}{D(72, 26, 35)}b_{72} & 0 \end{pmatrix},$$

$$S_3 := \begin{pmatrix} 0 & \frac{D(62,13,26)}{D(77,62,26)}b_{77} & D(71,13,35)b_{64} & D(77,64,15)b_{51} & D(77,64,51)b_{15} \\ \frac{D(62,13,26)}{D(77,62,26)}b_{77} & 0 & \frac{D(71,13,35)}{D(77,15,51)}b_{13} & \frac{D(77,13,62)}{D(77,62,26)}b_{26} & \frac{D(77,13,26)}{D(77,62,26)}b_{62} \\ D(71,13,35)b_{64} & \frac{D(71,13,35)}{D(77,15,51)}b_{13} & 0 & D(64,13,71)b_{35} & D(64,13,35)b_{71} \\ D(77,64,15)b_{51} & \frac{D(77,13,62)}{D(77,62,26)}b_{26} & D(64,13,71)b_{35} & 0 & \frac{D(77,64,15)D(51,26,35)}{D(44,26,35)}b_{44} \\ D(77,64,51)b_{15} & \frac{D(77,13,26)}{D(77,62,26)}b_{62} & D(64,13,35)b_{71} & \frac{D(77,64,15)D(51,26,35)}{D(44,26,35)}b_{44} & 0 \end{pmatrix},$$

$$S_4 := \begin{pmatrix} 0 & \frac{D(45,13,26)}{D(77,45,26)}b_{77} & D(56,13,35)b_{64} & D(77,64,32)b_{51} & D(77,64,51)b_{32} \\ \frac{D(45,13,26)}{D(77,45,26)}b_{77} & 0 & \frac{D(56,13,35)}{D(77,32,51)}b_{13} & \frac{D(77,13,45)}{D(77,45,26)}b_{26} & \frac{D(77,13,26)}{D(77,45,26)}b_{45} \\ D(56,13,35)b_{64} & \frac{D(56,13,35)}{D(77,32,51)}b_{13} & 0 & D(64,13,56)b_{35} & D(64,13,35)b_{56} \\ D(77,64,32)b_{51} & \frac{D(77,13,45)}{D(77,45,26)}b_{26} & D(64,13,56)b_{35} & 0 & \frac{D(77,64,32)D(51,26,35)}{D(63,26,35)}b_{63} \\ D(77,64,51)b_{46} & \frac{D(77,13,26)}{D(77,45,26)}b_{45} & D(64,13,35)b_{56} & \frac{D(77,64,32)D(51,26,35)}{D(63,26,35)}b_{63} & 0 \end{pmatrix}.$$

We can act by congruence on  $S_2$ ,  $S_3$  and  $S_4$  using the diagonal matrices:

$$N_2 := \text{diag} \left( \sqrt{A}, \sqrt{A} \frac{D(77,23,51)}{D(77,46,51)}, \frac{1}{\sqrt{A}} \frac{D(22,13,35)}{D(47,13,35)}, \frac{1}{\sqrt{A}} \frac{D(77,64,46)}{D(77,64,23)}, \frac{1}{\sqrt{A}} \right), \quad A := \frac{D(77,46,51)D(31,13,26)D(77,54,26)}{D(77,31,26)D(77,23,51)D(54,13,26)},$$

$$N_3 := \text{diag} \left( \sqrt{B}, \sqrt{B} \frac{D(77,15,51)}{D(77,46,51)}, \frac{1}{\sqrt{B}} \frac{D(22,13,35)}{D(71,13,35)}, \frac{1}{\sqrt{B}} \frac{D(77,64,46)}{D(77,64,15)}, \frac{1}{\sqrt{B}} \right), \quad B := \frac{D(77,46,51)D(77,62,26)D(31,13,26)}{D(62,13,26)D(77,15,51)D(77,31,26)},$$

$$N_4 := \text{diag} \left( \sqrt{C}, \sqrt{C} \frac{D(77,32,51)}{D(77,46,51)}, \frac{1}{\sqrt{C}} \frac{D(22,13,35)}{D(56,13,35)}, \frac{1}{\sqrt{C}} \frac{D(77,64,46)}{D(77,64,32)}, \frac{1}{\sqrt{C}} \right), \quad C = \frac{D(77,46,51)D(77,45,26)D(31,13,26)}{D(45,13,26)D(77,32,51)D(77,31,26)}.$$

Then  $S_1$ ,  $N_2S_2N_2$ ,  $N_3S_3N_3$  and  $N_4S_4N_4$  have the same entries on the common rows and columns; hence, we have the following  $8 \times 8$  symmetric matrix:

$$\begin{pmatrix} 0 & \frac{D(31,13,26)}{D(77,31,26)}b_{77} & D(22,13,35)b_{64} & D(77,64,46)b_{51} & D(77,64,51)b_{46} & D(77,64,51)b_{23} & D(77,64,51)b_{15} & D(77,64,51)b_{32} \\ * & 0 & \frac{D(22,13,35)}{D(77,46,51)}b_{13} & \frac{D(77,13,31)}{D(77,31,26)}b_{26} & \frac{D(77,13,26)}{D(77,31,26)}b_{31} & \frac{1}{A} \frac{D(31,13,26)D(77,13,26)}{D(54,13,26)D(77,31,26)}b_{54} & \frac{1}{B} \frac{D(31,13,26)D(77,13,26)}{D(62,13,26)D(77,31,26)}b_{62} & \frac{1}{C} \frac{D(31,13,26)D(77,13,26)}{D(45,13,26)D(77,31,26)}b_{45} \\ * & * & 0 & D(64,13,22)b_{35} & D(64,13,35)b_{22} & \frac{1}{A} \frac{D(22,13,35)D(64,13,35)}{D(47,13,35)}b_{47} & \frac{1}{B} \frac{D(22,13,35)D(64,13,35)}{D(71,13,35)}b_{71} & \frac{1}{C} \frac{D(22,13,35)D(64,13,35)}{D(56,13,35)}b_{56} \\ * & * & * & 0 & \frac{D(77,64,46)D(51,26,35)}{D(17,26,35)}b_{17} & \frac{1}{A} \frac{D(77,64,46)D(51,26,35)}{D(72,26,35)}b_{72} & \frac{1}{B} \frac{D(77,64,46)D(51,26,35)}{D(44,26,35)}b_{44} & \frac{1}{C} \frac{D(77,64,46)D(51,26,35)}{D(63,26,35)}b_{63} \\ * & * & * & * & 0 & X_{65}b_{65} & X_{53}b_{53} & X_{74}b_{74} \\ * & * & * & * & * & 0 & X_{36}b_{36} & X_{11}b_{11} \\ * & * & * & * & * & * & 0 & X_{27}b_{27} \\ * & * & * & * & * & * & * & 0 \end{pmatrix},$$

where the  $X_{ij}$  are to be determined in such a way that the rank of the matrix is equal to 4. For this purpose we note that we have determined the minors  $S_1$ ,  $N_2S_2N_2$ ,  $N_3S_3N_3$  and  $N_4S_4N_4$  by demanding precise relations among the eight vector columns  $V_i$  of the  $8 \times 8$  matrix  $\mathcal{M}$ . If we set:

$$\begin{aligned}
c_2 &:= \frac{D(46,64,51)D(77,31,26)}{D(31,13,26)}; & c_3 &:= \frac{D(77,46,51)}{D(22,13,35)}; \\
d_1 &:= \frac{1}{\sqrt{A}}; & d_2 &:= \frac{1}{\sqrt{A}} \frac{D(23,64,51)D(77,54,26)D(77,46,51)}{D(54,13,26)D(77,23,51)}; & d_3 &:= \sqrt{A} \frac{D(22,13,35)}{D(77,23,51)}; & d_4 &:= \sqrt{A} \frac{D(77,64,23)}{D(77,64,46)}; & d_6 &:= \sqrt{A}; \\
e_1 &:= \frac{1}{\sqrt{B}}; & e_2 &:= \frac{1}{\sqrt{B}} \frac{D(15,64,51)D(77,62,26)D(77,46,51)}{D(62,13,26)D(77,15,51)}; & e_3 &:= \sqrt{B} \frac{D(77,15,51)}{D(22,13,35)}; & e_4 &:= \sqrt{B} \frac{D(77,64,15)}{D(77,64,46)}; & e_7 &:= \sqrt{B}; \\
f_1 &:= \frac{1}{\sqrt{C}}; & f_2 &:= \frac{1}{\sqrt{C}} \frac{D(32,64,51)D(77,45,26)D(77,46,51)}{D(45,13,26)D(77,32,51)}; & f_3 &:= \sqrt{C} \frac{D(77,32,51)}{D(22,13,35)}; & f_4 &:= \sqrt{C} \frac{D(77,64,32)}{D(77,64,46)}; & f_8 &:= \sqrt{C};
\end{aligned}$$

then the following relations hold:

$$\begin{aligned}
V_1 + c_2 V_2 + c_3 V_3 + V_4 - V_5 &= 0, \\
d_1 V_1 + d_2 V_2 + d_3 V_3 + d_4 V_4 - d_6 V_6 &= 0, \\
e_1 V_1 + e_2 V_2 + e_3 V_3 + e_4 V_4 - e_7 V_7 &= 0, \\
f_1 V_1 + f_2 V_2 + f_3 V_3 + f_4 V_4 - f_8 V_8 &= 0,
\end{aligned}$$

each respectively on the rows and the columns of the corresponding  $5 \times 5$  minor.

In particular, the following relation holds on the first four rows:

$$\left(c_2 - \frac{d_2}{d_1}\right) V_2 + \left(c_3 - \frac{d_3}{d_1}\right) V_3 + \left(1 - \frac{d_4}{d_1}\right) V_4 - V_5 + \frac{d_6}{d_1} V_6 = 0.$$

Hence, it can be used to compute  $X_{65}$ , by demanding it on the fifth row:

$$X_{65}^{(5)} = \frac{1}{A} (A \cdot D(77, 23, 51) - D(77, 46, 51)) \frac{D(22, 31, 17)D(64, 13, 35)}{D(65, 31, 17)D(22, 13, 35)},$$

Otherwise we can compute  $X_{65}$  by demanding the relation on the sixth row, we get:

$$X_{65}^{(6)} = \left(\frac{1}{A} - \frac{D(77, 64, 23)}{D(77, 64, 46)}\right) \frac{D(77, 64, 46)D(51, 26, 35)D(72, 54, 47)}{D(72, 26, 35)D(65, 54, 47)}.$$

Writing the two formulas in terms of theta constants, we have

$$\begin{aligned}
X_{65}^{(5)} &= \pm \theta_{14} \theta_{33} \left( \frac{\theta_{00} \theta_{42} \theta_{57} \theta_{61} \theta_{70}}{\theta_{52} \theta_{75}} \right) \left( \frac{\theta_{02} \theta_{03} \theta_{24} \theta_{25}}{\theta_{41} \theta_{40} \theta_{66} \theta_{67}} - 1 \right), \\
X_{65}^{(6)} &= \pm \theta_{06} \theta_{21} \left( \frac{\theta_{00} \theta_{42} \theta_{57} \theta_{61} \theta_{70}}{\theta_{40} \theta_{67}} \right) \left( 1 - \frac{\theta_{03} \theta_{10} \theta_{24} \theta_{37}}{\theta_{52} \theta_{41} \theta_{66} \theta_{75}} \right).
\end{aligned}$$

By virtue of the Riemann relations in genus 3:

$$\theta_{52} \theta_{75} \theta_{41} \theta_{66} - \theta_{03} \theta_{10} \theta_{24} \theta_{37} = \theta_{14} \theta_{07} \theta_{33} \theta_{20}, \quad \theta_{40} \theta_{67} \theta_{41} \theta_{66} - \theta_{03} \theta_{02} \theta_{24} \theta_{25} = \theta_{06} \theta_{21} \theta_{07} \theta_{20}$$

the two expressions for  $X_{65}$  turn out to be equal:

$$X_{65}^{(5)} = \theta_{14} \theta_{33} \left( \frac{\theta_{00} \theta_{42} \theta_{57} \theta_{61} \theta_{70}}{\theta_{52} \theta_{75}} \right) \frac{\theta_{06} \theta_{21} \theta_{07} \theta_{20}}{\theta_{41} \theta_{40} \theta_{66} \theta_{67}} = \theta_{06} \theta_{21} \left( \frac{\theta_{00} \theta_{42} \theta_{57} \theta_{61} \theta_{70}}{\theta_{40} \theta_{67}} \right) \frac{\theta_{14} \theta_{07} \theta_{33} \theta_{20}}{\theta_{52} \theta_{41} \theta_{66} \theta_{75}} = X_{65}^{(6)}.$$



Likewise we get:

$$\begin{aligned}
X_{53} &= \frac{1}{B} (BD(77, 15, 51) - D(77, 46, 51)) \frac{D(22, 31, 17)D(64, 13, 35)}{D(53, 31, 17)D(22, 13, 35)}; \\
X_{74} &= \frac{1}{C} \left( 1 - C \frac{D(77, 64, 32)}{D(77, 64, 46)} \right) \frac{D(77, 64, 51)D(46, 31, 22)}{D(74, 31, 22)}; \\
X_{36} &= \frac{1}{B} \left( 1 - \frac{D(54, 13, 26)D(77, 23, 51)D(15, 64, 51)D(77, 62, 26)}{D(23, 64, 51)D(77, 54, 26)D(77, 15, 51)D(62, 13, 26)} \right) \frac{D(77, 64, 51)D(23, 47, 72)}{D(36, 47, 72)}; \\
X_{11} &= \left( \frac{1}{C} - \frac{D(77, 64, 32)}{AD(77, 64, 23)} \right) \frac{D(23, 54, 47)D(77, 64, 51)}{D(11, 54, 47)}; \\
X_{27} &= \left( \frac{1}{C} - \frac{D(77, 64, 32)}{BD(77, 64, 15)} \right) \frac{D(15, 62, 71)D(77, 64, 51)}{D(27, 62, 71)}.
\end{aligned}$$

Hence, each entry of the  $8 \times 8$  symmetric matrix with rank equal to 4 is uniquely determined, up to congruences by diagonal matrices. In particular, we will get a suitable form for the matrix we have determined, by multiplying it on both sides by the diagonal matrix  $\text{diag}(1, D(77, 31, 26), 1, 1, 1, 1, 1, 1)$ :

$$\begin{pmatrix}
0 & D(31, 13, 26)b_{77} & D(22, 13, 35)b_{64} & D(77, 64, 46)b_{51} & D(77, 64, 51)b_{46} & D(77, 64, 51)b_{23} & D(77, 64, 51)b_{15} & D(77, 64, 51)b_{32} \\
* & 0 & \frac{D(22, 13, 35)D(77, 31, 26)}{D(77, 46, 51)}b_{13} & D(77, 13, 31)b_{26} & D(77, 13, 26)b_{31} & \frac{1}{A} \frac{D(31, 13, 26)D(77, 13, 26)}{D(54, 13, 26)}b_{54} & \frac{1}{B} \frac{D(31, 13, 26)D(77, 13, 26)}{D(62, 13, 26)}b_{62} & \frac{1}{C} \frac{D(31, 13, 26)D(77, 13, 26)}{D(45, 13, 26)}b_{45} \\
* & * & 0 & D(64, 13, 22)b_{35} & D(64, 13, 35)b_{22} & \frac{1}{A} \frac{D(22, 13, 35)D(64, 13, 35)}{D(41, 13, 35)}b_{47} & \frac{1}{B} \frac{D(22, 13, 35)D(64, 13, 35)}{D(71, 13, 35)}b_{71} & \frac{1}{C} \frac{D(22, 13, 35)D(64, 13, 35)}{D(66, 13, 35)}b_{66} \\
* & * & * & 0 & \frac{D(77, 64, 46)D(51, 26, 35)}{D(17, 26, 35)}b_{17} & \frac{1}{A} \frac{D(77, 64, 46)D(51, 26, 35)}{D(17, 26, 35)}b_{17} & \frac{1}{B} \frac{D(77, 64, 46)D(51, 26, 35)}{D(44, 26, 35)}b_{44} & \frac{1}{C} \frac{D(77, 64, 46)D(51, 26, 35)}{D(63, 26, 35)}b_{63} \\
* & * & * & * & 0 & \frac{1}{A} (A \cdot D(77, 23, 51) - D(77, 46, 51)) \frac{D(22, 31, 17)D(64, 13, 35)}{D(65, 31, 17)D(22, 13, 35)}b_{65} & \frac{1}{B} (BD(77, 15, 51) - D(77, 46, 51)) \frac{D(22, 31, 17)D(64, 13, 35)}{D(53, 31, 17)D(22, 13, 35)}b_{53} & \frac{1}{C} \left( 1 - C \frac{D(77, 64, 32)}{D(77, 64, 46)} \right) \frac{D(77, 64, 51)D(46, 31, 22)}{D(74, 31, 22)}b_{74} \\
* & * & * & * & * & 0 & \frac{1}{B} \left( 1 - \frac{D(54, 13, 26)D(77, 23, 51)D(15, 64, 51)D(77, 62, 26)}{D(23, 64, 51)D(77, 54, 26)D(77, 15, 51)D(62, 13, 26)} \right) \frac{D(77, 64, 51)D(23, 47, 72)}{D(36, 47, 72)}b_{36} & \left( \frac{1}{C} - \frac{D(77, 64, 32)}{AD(77, 64, 23)} \right) \frac{D(23, 54, 47)D(77, 64, 51)}{D(11, 54, 47)}b_{11} \\
* & * & * & * & * & * & 0 & \left( \frac{1}{C} - \frac{D(77, 64, 32)}{BD(77, 64, 15)} \right) \frac{D(15, 62, 71)D(77, 64, 51)}{D(27, 62, 71)}b_{27} \\
* & * & * & * & * & * & * & 0
\end{pmatrix}$$

Using the expression of the Jacobian determinant in terms of theta constants and the Riemann relations we get the matrix  $L(\tau, z)$ :

$$L(\tau, z) = \begin{pmatrix}
0 & D(31, 13, 26)b_{77} & D(22, 13, 35)b_{64} & D(77, 64, 46)b_{51} & D(77, 64, 51)b_{46} & D(77, 64, 51)b_{23} & D(77, 64, 51)b_{15} & D(77, 64, 51)b_{32} \\
* & 0 & \pm \frac{\theta_{04}}{\theta_{06}} D(77, 31, 26)b_{13} & D(77, 13, 31)b_{26} & D(77, 13, 26)b_{31} & \pm \left( \frac{\theta_{07}\theta_{10}\theta_{13}\theta_{16}\theta_{19}}{\theta_{04}\theta_{40}\theta_{52}\theta_{67}\theta_{75}} \right) D(77, 23, 32)b_{54} & \pm \left( \frac{\theta_{07}\theta_{10}\theta_{13}\theta_{16}\theta_{19}}{\theta_{04}\theta_{43}\theta_{52}\theta_{67}\theta_{76}} \right) D(77, 15, 32)b_{62} & \pm \left( \frac{\theta_{04}\theta_{13}\theta_{16}\theta_{19}\theta_{23}}{\theta_{04}\theta_{40}\theta_{52}\theta_{67}\theta_{75}} \right) D(77, 23, 32)b_{45} \\
* & * & 0 & D(64, 13, 22)b_{35} & D(64, 13, 35)b_{22} & \pm \left( \frac{\theta_{04}\theta_{13}\theta_{16}\theta_{19}\theta_{23}}{\theta_{04}\theta_{40}\theta_{52}\theta_{67}\theta_{75}} \right) D(64, 23, 32)b_{47} & \pm \left( \frac{\theta_{04}\theta_{13}\theta_{16}\theta_{19}\theta_{23}}{\theta_{04}\theta_{41}\theta_{50}\theta_{67}\theta_{76}} \right) D(64, 15, 32)b_{71} & \pm \left( \frac{\theta_{04}\theta_{13}\theta_{16}\theta_{19}\theta_{23}}{\theta_{04}\theta_{42}\theta_{50}\theta_{67}\theta_{76}} \right) D(64, 23, 32)b_{66} \\
* & * & * & 0 & \pm \left( \frac{\theta_{04}}{\theta_{06}} \right) D(51, 26, 35)b_{17} & \pm \left( \frac{\theta_{04}\theta_{10}\theta_{13}\theta_{16}}{\theta_{41}\theta_{52}\theta_{60}\theta_{75}} \right) D(51, 23, 32)b_{72} & \pm \left( \frac{\theta_{04}\theta_{10}\theta_{13}\theta_{16}}{\theta_{41}\theta_{43}\theta_{50}\theta_{62}} \right) D(51, 15, 32)b_{44} & \pm \left( \frac{\theta_{04}\theta_{13}\theta_{16}\theta_{19}}{\theta_{43}\theta_{50}\theta_{60}\theta_{75}} \right) D(51, 23, 32)b_{63} \\
* & * & * & * & 0 & \pm \left( \frac{\theta_{04}\theta_{10}\theta_{13}\theta_{16}\theta_{19}}{\theta_{40}\theta_{41}\theta_{52}\theta_{60}\theta_{67}\theta_{75}} \right) D(22, 31, 17)b_{65} & \pm \left( \frac{\theta_{04}\theta_{10}\theta_{13}\theta_{16}\theta_{19}}{\theta_{41}\theta_{43}\theta_{50}\theta_{62}\theta_{67}\theta_{76}} \right) D(22, 31, 17)b_{53} & \pm \left( \frac{\theta_{04}\theta_{10}\theta_{13}\theta_{16}\theta_{19}}{\theta_{41}\theta_{43}\theta_{50}\theta_{60}\theta_{67}\theta_{76}} \right) D(46, 31, 22)b_{74} \\
* & * & * & * & * & 0 & \pm \left( \frac{\theta_{04}\theta_{10}\theta_{13}\theta_{16}\theta_{19}\theta_{23}}{\theta_{40}\theta_{41}\theta_{43}\theta_{50}\theta_{52}\theta_{60}\theta_{67}\theta_{75}\theta_{76}} \right) D(22, 31, 17)b_{36} & \pm \left( \frac{\theta_{04}\theta_{10}\theta_{13}\theta_{16}\theta_{19}\theta_{23}}{\theta_{40}\theta_{41}\theta_{43}\theta_{50}\theta_{52}\theta_{60}\theta_{67}\theta_{75}\theta_{76}} \right) D(77, 23, 32)b_{11} \\
* & * & * & * & * & * & 0 & \pm \left( \frac{\theta_{04}\theta_{10}\theta_{13}\theta_{16}\theta_{19}\theta_{23}}{\theta_{40}\theta_{41}\theta_{43}\theta_{50}\theta_{52}\theta_{60}\theta_{67}\theta_{75}\theta_{76}} \right) D(22, 31, 17)b_{27} \\
* & * & * & * & * & * & * & 0
\end{pmatrix}$$

**Remark 6.1.** Note that each coefficient can be written as a product of at most 8 determinants over 7 determinants, although there seems not to be any canonical choice for such an expression.

Take, for instance, the coefficient  $X_{65}$ ; the triples of even characteristics  $\{(06), (07), (14)\}$  and  $\{(40), (41), (52)\}$  extend to azygetic 5-tuples by means of the same pair  $\{(55), (70)\}$ , and the triples  $\{(20), (21), (33)\}$  and  $\{(66), (67), (75)\}$  extend to azygetic 5-tuples by means of the pair  $\{(34), (70)\}$ . Therefore, we can write:

$$X_{65} = \pm \frac{\theta_{55}\theta_{70}}{\theta_{55}\theta_{70}} \left( \frac{\theta_{06}\theta_{07}\theta_{14}}{\theta_{40}\theta_{41}\theta_{52}} \right) \cdot \frac{\theta_{34}\theta_{70}}{\theta_{34}\theta_{70}} \left( \frac{\theta_{20}\theta_{21}\theta_{33}}{\theta_{66}\theta_{67}\theta_{75}} \right) D(22, 31, 17) = \pm \frac{D(11, 53, 72)D(11, 13, 74)}{D(11, 15, 72)D(11, 13, 32)} D(22, 31, 17).$$

In a similar way, we can get such an expression for  $X_{13}$  and  $X_{53}$

$$X_{13} = \pm \frac{\theta_{60}}{\theta_{04}} D(77, 31, 26) = \frac{D(22, 13, 35)}{D(77, 46, 51)} D(77, 31, 26),$$

$$X_{53} = \pm \frac{\theta_{37}\theta_{61}}{\theta_{37}\theta_{61}} \left( \frac{\theta_{05}\theta_{07}\theta_{21}}{\theta_{41}\theta_{43}\theta_{67}} \right) \cdot \frac{\theta_{37}\theta_{70}}{\theta_{37}\theta_{70}} \left( \frac{\theta_{14}\theta_{16}\theta_{30}}{\theta_{50}\theta_{52}\theta_{76}} \right) D(22, 31, 17) = \pm \frac{D(26, 36, 65)D(26, 27, 74)}{D(23, 26, 36)D(26, 27, 32)} D(22, 31, 17).$$

and so on for each entry of the matrix.

We can summarize the previous discussion in the following way.

**Theorem 6.2.** *Let  $\tau$  be the period matrix of the jacobian of a smooth plane quartic. When an even characteristic and a corresponding Aronhold set of characteristics (i.e. a level 2 structure) are fixed then, up to congruences, a unique matrix  $L(\tau, z)$  of rank four is determined in such a way that its entries are proportional to the 28 bitangents. The equation of the corresponding plane quartic is obtained taking the determinant of any minor of degree four of the above matrix.*

Hence for determining the matrix  $A(z)$  we can consider the minor obtained taking the first 4 rows and columns of the matrix  $L(\tau, z)$  divided for a suitable jacobian determinant, so we will get modular functions as coefficients, as stated in the following corollary.

**Corollary 6.3.** *Let  $\tau$  be the period matrix of the jacobian of a smooth plane quartic, then the matrix  $A(z)$  is congruent to the following matrix:*

$$Q(\tau, z) = \begin{pmatrix} 0 & \frac{D(31,13,26)}{D(77,31,26)} b_{77} & \frac{D(22,13,35)}{D(77,31,26)} b_{64} & \frac{D(77,64,46)}{D(77,31,26)} b_{51} \\ * & 0 & \frac{D(22,13,35)}{D(77,46,51)} b_{13} & \frac{D(77,13,31)}{D(77,31,26)} b_{26} \\ * & * & 0 & \frac{D(64,13,22)}{D(77,31,26)} b_{35} \\ * & * & * & 0 \end{pmatrix}.$$

Moreover

$$\det Q(\tau, z) = 0,$$

is an equation for the plane quartic.

A very similar equation for the plane quartic has been already obtained in [Gu11]. This is a classical result that goes back to Riemann; we refer to [Do12] for details.

## REFERENCES

- [CS03] L. Caporaso, E. Sernesi, *Recovering plane curves from their bitangents*. Journal of Algebraic Geometry 12: 225-244, 2003.
- [CS03b] L. Caporaso, E. Sernesi, *Characterizing curves by their odd theta-characteristics*. Reine Angew. Math. 562: 101-135, 2003.
- [Do12] I. Dolgachev, *Classical Algebraic Geometry: a Modern View*. Cambridge Univ. Press, 2012.
- [DO88] I. Dolgachev, D. Ortland *Point Sets in Projective Spaces and Theta Functions*. Société Mathématique de France, 1988.

- [Fay79] J. Fay, *On the Riemann-Jacobi formula*. Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, (5):61–73, 1979.
- [vGvdG86] B. van Geemen and G. van der Geer, *Kummer varieties and the moduli spaces of abelian varieties*. Amer. J. Math, 108(3):615–641, 1986.
- [GSM04] S. Grushevsky and R. Salvati Manni, *Gradients of odd theta functions*. Reine Angew. Math. 573, 45–59, 2004.
- [Gu11] J. Guàrdia, *On the Torelli problem and Jacobian nullwerte in genus three*. Mich. Math. J. 60, No. 1, 51–65, 2011.
- [H55] O. Hesse, *Über die Doppeltangenten der Curven vierter Ordnung*. J. Reine Angew. Math. 49, 279–332, 1855.
- [Igu72] J.-I. Igusa, *Theta functions*. Grundlehren der Mathematischen Wissenschaften, Volume 194 Springer-Verlag, New York, 1972.
- [Igu80] J.-I. Igusa, *On Jacobi’s derivative formula and its generalizations*. Amer. J. Math, 102(2):409–446, 1980.
- [Igu81] J.-I. Igusa, *On the Nullwerte of Jacobians of odd theta functions*. Algebraic geometry, int. Symp. Centen. Birth F. Severi, Roma 1979, Symp. Math. 24, 83–95, 1981.
- [Igu83] J.-I. Igusa, *Multiplicity one theorem and problems related to Jacobi’s formula*. Amer. J. Math. 105:409–446, 157–187, 1983.
- [L05] D. Lehavi, *Any smooth plane quartic can be reconstructed from its bitangents*. Israel Journal of Mathematics, 146 (1), 371–379, 2005.
- [PSV11] D. Plaumann, B. Sturmfels and C. Vinzant, *Quartic curves and their bitangents*. Symb. Comput. 46, 712–733, 2011.
- [RF74] H. Rauch and H. Farkas, *Theta functions with applications to Riemann surfaces*. The Williams & Wilkins Co., Baltimore, Md., 1974.
- [SM83] R. Salvati Manni, *On the nonidentically zero Nullwerte of Jacobians of theta functions with odd characteristics*. Adv. in Math, 47(1):88–104, 1983.
- [SM85] R. Salvati Manni, *On the dimension of the vector space  $\mathbb{C}[\theta_m]_4$* . Nagoya Math. J, 98:99–107, 1985.
- [SM94] R. Salvati Manni, *Modular varieties with level 2 theta structure*. Amer. J. Math. 116(6):1489–1511, 1994.

UNIVERSITÀ “LA SAPIENZA”, DIPARTIMENTO DI MATEMATICA, PIAZZALE A. MORO 2, I-00185, ROMA, ITALY  
*E-mail address:* dallapiazza@mat.uniroma1.it

*E-mail address:* fiorentino@mat.uniroma1.it

*E-mail address:* salvati@mat.uniroma1.it